Solving Rubik's Cube

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Abstract. This document contains a step-by-step solution of Rubik's Cube. Although brief, the introduction contains some small extra information about the history and the relationship between the cube and group theory with some references for those interested.

Figure 1. Ernö Rubik, creator of the Rubik's Cube.

1. Introduction

Rubik's Cube was invented by Ernö Rubik, the guy in figure 1, in the spring of 1974 in his home town of Budapest, Hungary. It is said that his inspiration for the cube's internal mechanism came from pebbles in the River Danube whose edges had been smoothed away. Rubik called his invention the **magic cube** and it was renamed the **Rubik's Cube** by the Ideal Toy Corporation in 1980.

The solution of the cube I will present here was not found by me. However, I do not have the original reference for it once it was taught to me by a friend that learned from someone else. As I have learned the solution while preparing a coursework for a Group Theory course, it is worth saying something about this connection.

The group theory inspired by the cube is huge. The allowed twists of the cube obey in a very straightforward way the requirements to form a group and there are a lot of interesting curiosities about them. For the interested reader, the best references are probably Hofstadter's *Metamagical Themas*[1], which has a popular level treatment of the subject, and Singmaster[2] for a more technical and detailed exposition.

Although I did not explore it very much yet, http://www.rubiks.com/ seems to be a nice site about the cube with a lot of information. It was from there that took the information in the first paragraph. The Wikipedia article at http://en.wikipedia.org/wiki/Rubik's_Cube is also a good reference with a list of links in the end.

2. Notation

Figure 2 shows a 3×3 cube. Each little square inside each face of a cube is called a **cubicle** and an $n \times n$ cube has n^2 cubicles in each face. The 3×3 cube is the original model designed by Rubik and the most popular of all. Let us name the faces of the cube in order to be able to write down its solution. Each face receives a name which will be

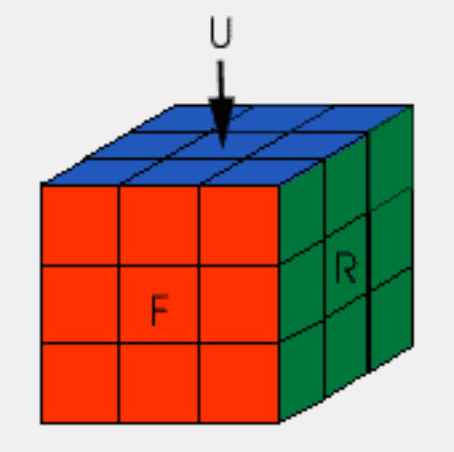

Figure 2. Three of the six faces of the cube: F = Front, U = Up and R = Right.

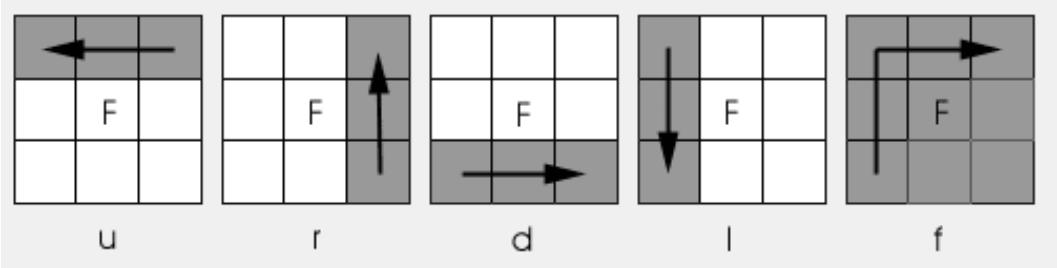

Figure 3. The five basic general moves of the cube.

written as a capital roman letter. Holding the cube in front of your eyes, the face you will be looking at will be called F (for Front). The R (Right) and U (Up) faces are as drawn in figure 2. The opposite face to F is called B (Back), the opposite to U is D (Down) and the opposite to R is L (Left).

An $n \times n$ cube has n horizontal layers and n vertical layers considering we are looking at the F face. We will give special names to the horizontal ones. The uppermost is called the **top layer** and the down most the **bottom layer**. In the case of the 3×3 cube, the layer in between these two is called the **middle layer**.

The moves we are going to need in order to solve the cube form a set of six basic moves that are indicated by small roman letters. The first five are shown in figure 3. Each one of them consists of a 90° turning of the cubicles painted in gray in the direction indicated by the arrow. The moves are always described considering that you are holding the cube in front of your eyes. A prime (') after the move indicates the move in the opposite direction, for example, while f indicates a turning of the F face clockwise, f' will indicate the turning of the same face counterclockwise. Although in the figure they are depicted in a 3×3 cube, these five moves are basically the same for any $n \times n$ cube.

The sixth move will be defined only for the 3×3 cube and consists in turning the middle layer of the F face by 90°, leaving the upper and bottom layers unmoved. It is

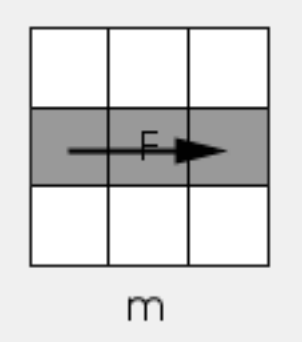

Figure 4. The m move, only defined in the 3×3 cube.

symbolised by the letter m and depicted in figure 4.

A sequence of basic moves in a certain order is called an **operator**. Although we will change this convention afterwards, throughout this chapter we read the moves in an operator from left to right. This means that the operator $rfr'd$, for instance, represent the combined action of first doing the move r, then f, then r in the opposite direction and finally d.

If a move or a sequence of moves need to be repeated a certain number of times, we will indicate it by a power. For example r^3 indicate the sequence of moves rrr while $(rd)^4$ indicates $rdrdrdrd$.

With these definitions, we can finally pass to a simple method of solution of the entire 3×3 cube.

3. Solution Method

The solution of the cube to be presented here has three phases. In each phase, one entire layer of the cube is solved. All that is needed is a set of appropriate operators. Actually, with some practice and by memorizing this small set, the complete cube can be reassembled from any initial configuration.

3.1. First Phase: Top Layer

The first task is to solve one face of the cube such that the laterals of that face are matching correctly the center cubicles of the adjacent faces. The colored cubicles figure 5 show the important cubicles that must be matched at the F and R faces if we consider that the solved face is U. The same must apply to B and L as well. There is no need to worry about the D face at this stage. The solution of one face is not difficult. If you still do not know how to do it, it is an excellent exercise to get acquainted to the cube and acquire some practice before start to solve it all.

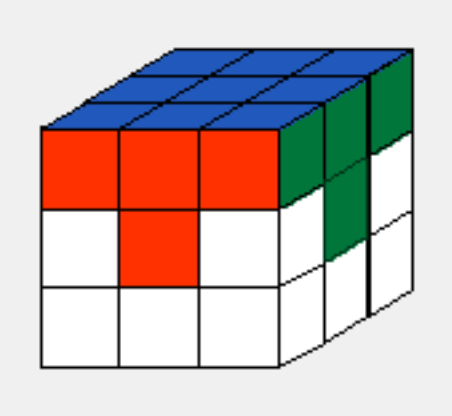

Figure 5. Example of solved U face with lateral cubicles matching the center cubicles of the adjacent faces.

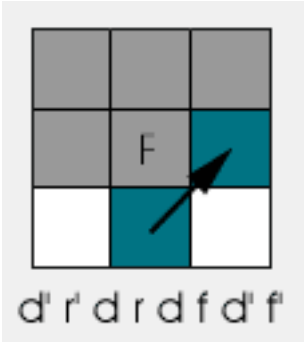

Figure 6. $d'r'drdfd'f'$

3.2. Second Phase: Middle Layer

The second phase consists in solving the middle layer of the cube. To do this we will only need to two operators. The important operators are $d'r'drdfd'f'$ and $dld'l'd'f'df$ and their result is depicted respectively in figures 6 and 7. These moves will move the green cubicle at the start of the arrow to the place at the end of the arrow. They will let the gray painted cubicles and the entire top layer intact. The move can be used either to put a cubicle in the right place or to take it out from a wrong place, for the cubicle at the end of the arrow will be put in another cubicle in the bottom layer of the cube, the exact place is not important. With these simple operators it is possible to solve the middle layer and have both the upper and the middle layers of the cube completed. However, to solve the last layer we will need more complicated operators.

3.3. Last Phase: Bottom Layer

It is possible that after the second phase you will already have a solved cube in your hands. Although this can happen, it is highly unprobable and some more moves will be necessary to finish the solution. To the last step then we need to turn the whole cube upside down such that the unsolved face will now be the U face. To finish solving the

Figure 7. $dld'l'd'f'df$

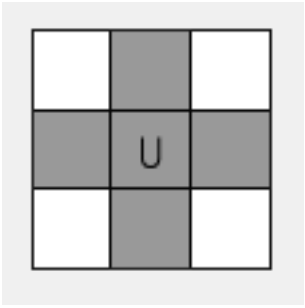

Figure 8. The middle cross of the U face that is to be solved first.

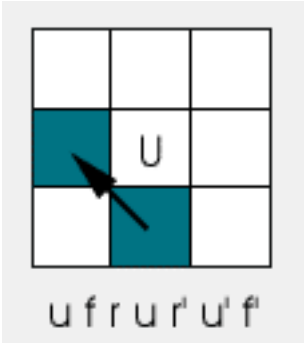

Figure 9. $ufrur'u'f'$

cube we will need to know four more operators.

First, we will solve the middle cross of the U face, which is indicated in figure 8. We will not bother about the corners right now. In order to solve these cubicles we use two new operators. The first one will be used to bring the corresponding cubicles of the middle cross to the correct places, but they may reach their positions twisted. The second operator will then be used to untwist these cubicles.

The operator used to align the middle cross is shown in figure 9. It is given by the sequence $ufrur'u'f'$. This operator will change the indicated cubicles without misaligning the other two, but may twist them. We will be able to fix this afterwards.

Figure 10 shows an example of configuration of the cube where the middle cross has

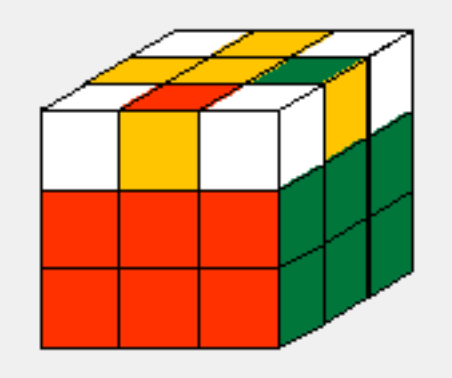

Figure 10. Example of a resulting configuration with two twisted faces in the middle cross.

all cubicles aligned correctly, but two of them are twisted. The interesting feature when you align the middle cross is that in every resulting configuration the twisted cubicles come in pairs. You will never see one or three cubicles twisted, only two or four. The next operator will untwist them.

Now you must pay attention because this is a very tricky operator. The operator works in this way: consider that the F face is in the bottom of figure 11 and you are looking at the cube from above, i.e., looking at the U face. You must put the cubicle you want to twist in the position marked by the small circle with an arrow. Then, you will do the first part of the operator, namely the $(rm)^4 = rmrmrmrm$ sequence of moves. Note that although the figure represents an upper vision of the cube, the moves are ALWAYS defined as if you are looking at the F face!

Be very, very careful now! The "$\circ$" between the two sequences of movements indicates that you will need to turn the U face WITHOUT TURNING THE REST OF THE CUBE until the next cubicle you want to untwist reaches the position indicated with the small circle with the arrow. Do not bother if your cube seems to be a mess at this point, when you finish the next four rm moves you will find your cube in the correct configuration. The entire operator is given by $(rm)^4 \circ (rm)^4$. If you have four cubicles to untwist, just repeat this operator with the two remaining twisted cubicles one more time.

At this point we are only two operators away from solving the entire cube. The next stage is to put the corners of the U face in the right place, although they may end up twisted. The operator needed to do that is shown in figure 12 and is given by $lu'r'ul'u'ru$. It will change the places of the cubicles as shown by the arrows without affecting the already solved middle cross (and obviously the rest of the cube).

You will use the next operator until your cube is totally solved. Do not stop and pay a lot of attention, because if you do something wrong, you will probably have to start it all again once that your cube will be totally scrambled. The operator $rf'r'f$, shown in figure 13 twists only the indicated corner. You will have to use this operator repeatedly as many times as needed until the corner you are trying to untwist reaches

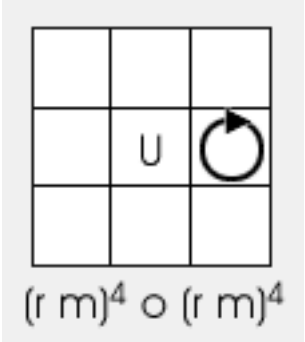

Figure 11. $(rm)^4 \circ (rm)^4$

Figure 12. $lu'r'ul'u'ru$

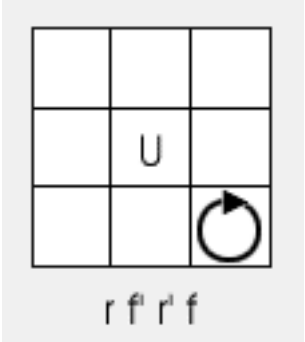

Figure 13. $rf'r'f$

the correct orientation. The cube will look a complete mess when this happens, but the corner will be correct.

Do not panic, but do not move! After untwisting that corner, moving only the top layer, put the next corner to be untwisted in the previously indicated position and repeat the same operator until this cubicle becomes totally untwisted too. You will need to keep doing this flawlessly until all the corners are untwisted. Amazingly, when you untwist the last corner, the cube will be then solved!

References

[1] Hofstadter, D. R. *Metamagical Themas: Questing for the Essence of Mind and Pattern.* HarperCollins Publishers, May (1996).

[2] Singmaster, D. *Notes on Rubik's "Magic Cube".* Penguin Books Ltd, September (1981).